

Nonlinear hidden symmetries in General Relativity and String Theory: a matrix generalization of the Ernst potentials

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Abstract

In this paper we recall a simple formulation of the stationary electrovacuum theory in terms of the famous complex Ernst potentials, a pair of functions which allows one to generate new exact solutions from known ones by means of the so-called nonlinear hidden symmetries of Lie–Bäcklund type. This formalism turned out to be very useful to perform a complete classification of all 4D solutions which present two spacetime symmetries or possess two Killing vectors. Curiously enough, the Ernst formalism can be extended and applied to stationary General Relativity as well as the effective heterotic string theory reduced down to three spatial dimensions by means of a (real) matrix generalization of the Ernst potentials. Thus, in this theory one can also make use of nonlinear matrix hidden symmetries in order to generate new exact solutions from seed ones. Due to the explicit independence of the matrix Ernst potential formalism of the original theory (prior to dimensional reduction) on the dimension D , in the case when the theory initially has $D \geq 5$, one can generate new solutions like *charged* black holes, black rings and black Saturns, among others, starting from uncharged field configurations.

1 Introduction

It is well known that both, the stationary action and the coupled field equations of the Einstein–Maxwell theory can be formulated in terms of a pair of very simple complex

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functions that were called *Ernst potentials* after their inventor [1, 2]. In the language of these potentials, the black holes of Schwarzschild and Kerr, Reissner–Nordström and Kerr–Newmann adopt a very simple form, as well as some cosmological models, among other exact solutions [1, 3]. Indeed, this formalism facilitates the general study of the symmetries of the theory and, hence, the construction of new exact solutions by means of very well-known solution-generating techniques (see, for instance, [4]).

It turns out that the Ernst formalism can be generalized to low-energy effective string theories and General Relativity with extra dimensions in terms of matrix potentials instead of complex functions (see [3],[5]–[7], for instance). This matrix formalism also enables one to study the complete symmetry group of the underlying theory and to apply generalized solution-generating techniques with matrix charges involved [8]–[9]. In particular, this matrix formalism can be applied to the classification and construction of charged black holes, black rings and black Saturns in 5D and multiple black rings in $D \geq 6$ in the framework of such theories [10]–[11].

In this paper we first recall the derivation of the Ernst potentials for the stationary Einstein–Maxwell theory and write both field equations and the effective action in their language. We further refer to the stationary formulation of the low-energy heterotic string theory, and the corresponding field equations, in terms of a pair of matrix Ernst potentials that closely resembles the formulation of the stationary theory of electrovacuum in the language of the complex Ernst potentials. A fact that, in principle, allows one to generalize all the so far obtained results in the stationary Einstein–Maxwell theory to the realm of the stationary heterotic string theory.

As an extra bonus, within the framework of higher dimensional General Relativity and the low energy limit of heterotic string theory, the matrix Ernst potentials can be used to classify and construct exact solutions that corresponds to higher dimensional objects like black holes, black rings, black Saturns and multiple black rings. A sketch of how this program can be performed is given at the end of this paper.

2 Ernst potentials in the stationary Einstein–Maxwell theory

In this section we briefly review the derivation of the Ernst potentials within the framework of the stationary Einstein–Maxwell theory basically following the work given by [2].

Let us consider the 4D action of the electrovacuum theory

$$\mathcal{S}_{EM} = \int d^4x \sqrt{|G|} \left({}^4R - \frac{1}{4} F_{mn}^2 \right), \quad (1)$$

where G is the determinant of the metric G_{mn} , $F_{mn} = \partial_m A_n - \partial_n A_m$, A_m is the gauge field, 4R is the scalar curvature in 4D and $m, n, = 0, 1, 2, 3$; $\mu, \nu = 1, 2, 3$.

Consider now the stationary ansatz for the metric

$$ds^2 = G_{mn} dx^m dx^n = -f(dt + \omega_\mu dx^\mu)^2 + f^{-1} \gamma_{\mu\nu} dx^\mu dx^\nu, \quad (2)$$

where f , $\gamma_{\mu\nu}$ and ω_μ are quantities independent on t .

Indices of spatial coordinates are raised and lowered with the aid of the metric tensor $\gamma_{\mu\nu}$ and its inverse $\gamma^{\mu\nu}$, unless otherwise indicated through a left superindex $^{(0)}$.

Thus, if F_{mn} is a covariant tensor, then

$$F^{\alpha\beta} = \gamma^{\alpha\mu}\gamma^{\beta\nu}F_{\mu\nu} \quad \text{and} \quad {}^{(0)}F^{\alpha\beta} = g^{\alpha m}g^{\beta n}F_{mn}.$$

The three-dimensional vector ω_μ can always be dualized through an invariant torsion vector in the following form

$$f^{-2}\tau^\mu = -\gamma^{-1/2}\epsilon^{\mu\rho\sigma}\partial_\rho\omega_\sigma \quad (3)$$

or, equivalently,

$$f^{-2}\vec{\tau} = -\nabla \times \vec{\omega}, \quad (4)$$

by making use of the three-dimensional vectorial calculus which employs $\gamma_{\mu\nu}dx^\mu dx^\nu$ as background metric.

Let us now consider a stationary electromagnetic field $F_{mn} = \partial_m A_n - \partial_n A_m$ with the given metric.

The stationarity condition $\partial_0 A_m = 0$ for the electric field implies

$$F_{0\nu} = -\partial_\nu A_0, \quad (5)$$

while the sourceless Maxwell equations

$$\partial_\nu \left[(-g)^{1/2} {}^{(0)}F^{m\nu} \right] = 0 \quad (6)$$

in the case when $m = \mu$ provide us with the magnetic components

$${}^{(0)}F^{\mu\nu} = f\gamma^{-1/2}\epsilon^{\mu\nu\rho}\partial_\rho\psi, \quad (7)$$

in terms of the scalar magnetic potential ψ .

It turns out that all the remaining components can be expressed as functions of these six magnitudes; for instance,

$${}^{(0)}F^{0\nu} = \omega_\mu^{(0)}F^{\mu\nu} + \gamma^{\mu\nu}F_{0\mu}, \quad (8)$$

is an identity that is directly inferred from the stationary metric.

By substituting the relations (8), (7), (5) and (3) in the Maxwell equations (6) with $m = 0$ one gets

$$\nabla \left(f^{-1}\nabla A_0 \right) = -f^{-2}\vec{\tau} \cdot \nabla\psi. \quad (9)$$

By rewriting $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ with the aid of the relations (5) and (7), and making use of the expression for the cyclic identity $\epsilon^{\mu\nu\rho}\partial_\rho F_{\mu\nu} = 0$, one obtains

$$\nabla \left(f^{-1}\nabla\psi \right) = f^{-2}\vec{\tau} \cdot \nabla A_0. \quad (10)$$

Now one is able to introduce the scalar complex potential

$$\Phi = A_0 + i\psi, \quad (11)$$

which is precisely the electromagnetic Ernst potential.

By combining (9) and (10) one obtains a single complex equation

$$\nabla \left(f^{-1} \nabla \Phi \right) = i f^{-2} \vec{\tau} \cdot \nabla \Phi. \quad (12)$$

Thus, in this way we have reduced the stationary Maxwell equations to a single equation in terms of the complex electromagnetic Ernst potential.

On the other side, within the framework of the Einstein equations for the gravitational field, it turns out convenient to express the Ricci tensor

$$R_{mn} = \partial_m \Gamma_{na}^a - \partial_a \Gamma_{mn}^a + \Gamma_{bm}^a \Gamma_{an}^b - \Gamma_{ba}^a \Gamma_{mn}^b \quad (13)$$

in terms of a complex three-dimensional vector \vec{G} defined by

$$2f\vec{G} = \nabla f + i\vec{\tau} \quad (14)$$

for the general case of the stationary metric.

In this way we can obtain the following relations

$$-f^{-2} R_{00} = \nabla \vec{G} + (\vec{G}^* - \vec{G}) \cdot \vec{G}, \quad (15)$$

$$-2if^{-2} {}^{(0)}R_0^\mu = \gamma^{-1/2} \epsilon^{\mu\rho\sigma} (\partial_\sigma G_\rho + G_\rho G_\sigma^*), \quad (16)$$

$$f^{-2} (\gamma_{\rho\mu} \gamma_{\sigma\nu} {}^{(0)}R^{\mu\nu} - \gamma_{\rho\sigma} R_{00}) = R_{\rho\sigma}(\gamma) + G_\rho G_\sigma^* + G_\rho^* G_\sigma, \quad (17)$$

where $R_{\rho\sigma}(\gamma)$ stands for the Ricci tensor calculated through the three-dimensional metric $\gamma_{\mu\nu} dx^\mu dx^\nu$.

Thus, from the above obtained formulas, for the energy-momentum tensor of the electromagnetic field

$$-4\pi T_{mn} = g^{ab} F_{ma} F_{nb} - \frac{1}{4} g_{mn} F_{ab} F^{ab} \quad (18)$$

one gets the following relations

$$\frac{1}{2} F_{mn} F^{mn} = (\nabla \psi)^2 - (\nabla A_0)^2, \quad (19)$$

$$8\pi f^{-1} T_{00} = (\nabla \psi)^2 + (\nabla A_0)^2, \quad (20)$$

$$4\pi f^{-1(0)}T_0^\mu = \gamma^{-1/2}\epsilon^{\nu\rho\sigma}(\partial_\rho\psi)(\partial_\sigma A_0), \quad (21)$$

$$-4\pi f^{-1(0)}T^{\mu\nu} = (\partial^\mu\psi)(\partial^\nu\psi) + (\partial^\mu A_0)(\partial^\nu A_0) - \frac{1}{2}\gamma^{\mu\nu}[(\nabla\psi)^2 + (\nabla A_0)^2], \quad (22)$$

where $\partial^\mu = \gamma^{\mu\nu}\partial_\nu$.

By making use of the Einstein equations

$$R_{mn} = -8\pi T_{mn}, \quad (23)$$

from the relations (16) and (21) one obtains

$$\nabla \times \vec{\tau} = -4\nabla\psi \times \nabla A_0 = i\nabla \times (\Phi\nabla\Phi^* - \Phi^*\nabla\Phi). \quad (24)$$

In this way, the following equation

$$\vec{\tau} + i(\Phi^*\nabla\Phi - \Phi\nabla\Phi^*) = \nabla\chi \quad (25)$$

defines the scalar potential χ up to an additive constant.

Now let us define the complex scalar potential

$$E = f - \Phi\Phi^* + i\chi, \quad (26)$$

called gravitational Ernst potential.

This potential allows one to obtain, from the relations (14) and (25), the following equality

$$f\vec{G} = \frac{1}{2}\nabla E + \Phi^*\nabla\Phi. \quad (27)$$

By substituting (27) in the gravitational field equations (15) and (20), and making use of the Maxwell equations (12), we obtain a single equation

$$f\nabla^2 E = (\nabla E + 2\Phi^*\nabla\Phi) \cdot \nabla E; \quad (28)$$

on the other hand, the relation (12) can be expressed in the following way:

$$f\nabla^2\Phi = (\nabla E + 2\Phi^*\nabla\Phi) \cdot \nabla\Phi. \quad (29)$$

It is evident that from the definition (26), one can obtain the following expression for the function f :

$$f = \frac{1}{2}(E + E^*) + \Phi\Phi^*. \quad (30)$$

Thus, relations (28) and (29) are the well-known differential Ernst equations for the stationary electrovacuum.

Finally, the gravitational field equations (17) and (22) reduce to the following expression

$$-f^2 R_{\mu\nu} = \frac{1}{2} E_{,(\mu} E_{,\nu)}^* + \Phi E_{,(\mu} \Phi_{,\nu)}^* + \Phi^* E_{,(\mu} \Phi_{,\nu)} - (E + E^*) \Phi_{,(\mu} \Phi_{,\nu)}^*, \quad (31)$$

where the symmetrization of indices are defined in the following form

$$2E_{,(\mu} E_{,\nu)}^* \equiv (\partial_\mu E)(\partial_\nu E^*) + (\partial_\nu E)(\partial_\mu E^*). \quad (32)$$

In this way, the field equations for the Ernst potentials (28) and (29), together with the Einstein equations (31), determine the dynamics of the field system of the stationary Einstein–Maxwell theory.

This system of self-consistent second order differential equations, despite their apparent simplicity, has no general solution at the moment. Only particular solutions are known in the literature and it is of great relevance to obtain new solutions possessing a coherent and consistent physical interpretation. It is worth noticing that precisely at this point is where the solution-generating techniques (which make use of nonlinear hidden symmetries to construct new solutions starting from seed ones) can be of great help towards this aim.

2.1 Effective action of the stationary EM theory and Ernst potentials

Now let us express the effective action of the stationary Einstein–Maxwell theory from which one can derive both the Einstein equations (31), and the Ernst equations (28) and (29) by the variational method.

By redefining the electromagnetic Ernst potential as follows

$$\Phi \equiv \frac{1}{\sqrt{2}} F, \quad (33)$$

the effective stationary action of the Einstein–Maxwell theory adopts the following form

$${}^4\mathcal{S}_{EM} = \int d^3x \, |g|^{\frac{1}{2}} \left(-{}^3R + {}^3\mathcal{L}_{EM} \right),$$

where the matter Lagrangian ${}^3\mathcal{L}_{EM}$ is given by

$${}^3\mathcal{L}_{EM} = \frac{1}{2f^2} |\nabla E + F^* \nabla F|^2 - \frac{1}{f} |\nabla F|, \quad (34)$$

where now $f = \frac{1}{2}(E + E^* + FF^*)$. It is a straightforward exercise to vary this action and obtain the above quoted Einstein and Ernst equations.

3 Low energy effective action of heterotic string and matrix Ernst potentials

The effective action of the low-energy limit of the heterotic string at tree level takes into account just the massless modes of the theory and possesses the form [12, 13]

$$\mathcal{S}^{(D)} = \int d^{(D)}x \sqrt{|G^{(D)}|} e^{-\phi^{(D)}} \left(R^{(D)} + \phi_{;M}^{(D)} \phi^{(D);M} - \frac{1}{12} H_{MNP}^{(D)} H^{(D)MNP} - \frac{1}{4} F_{MN}^{(D)I} F^{(D)IMN} \right), \quad (35)$$

where

$$F_{MN}^{(D)I} = \partial_M A_N^{(D)I} - \partial_N A_M^{(D)I}, \quad I = 1, 2, \dots, n;$$

$$H_{MNP}^{(D)} = \partial_M B_{NP}^{(D)} - \frac{1}{2} A_M^{(D)I} F_{NP}^{(D)I} + \text{cyclic perms. of } M, N \text{ and } P.$$

Here $G_{MN}^{(D)}$ is the metric, $B_{MN}^{(D)}$ is the anti-symmetric Kalb-Ramond tensor field, $\phi^{(D)}$ is the dilaton and $A_M^{(D)I}$ is a set of $U(1)$ vector fields ($I = 1, 2, \dots, n$). D is the dimensionality of the spacetime and $M, N, P = 1, 2, 3, \dots, 10$. In the consistent critical case (where the quantum theory is free of anomalies) $D = 10$ and $n = 16$, but we shall leave these parameters arbitrary in our analysis for the sake of generality.

By following Maharana and Schwarz [12], and Sen [13], we further perform the dimensional reduction of this model on a $D-3 = d$ -torus. Thus, the resulting three-dimensional, stationary theory possesses the $SO(d+1, d+1+n)$ symmetry group and describes gravity in terms of the metric tensor

$$g_{\mu\nu} = e^{-2\phi} \left(G_{\mu\nu}^{(D)} - G_{p+3,\mu}^{(D)} G_{q+3,\nu}^{(D)} G^{pq} \right), \quad (36)$$

where the subscripts $p, q = 1, 2, \dots, d$; coupled to the following set of three-dimensional fields:

a) scalar fields

$$G = \left(G_{pq} = G_{p+3,q+3}^{(D)} \right), \quad B = \left(B_{pq} = B_{p+3,q+3}^{(D)} \right),$$

$$A = \left(A_p^I = A_{p+3}^{(D)I} \right), \quad \phi = \phi^{(D)} - \frac{1}{2} \ln |\det G|. \quad (37)$$

b) antisymmetric tensor field of second rank

$$B_{\mu\nu} = B_{\mu\nu}^{(D)} - 4B_{pq} A_\mu^p A_\nu^q - 2 \left(A_\mu^p A_\nu^{p+d} - A_\nu^p A_\mu^{p+d} \right), \quad (38)$$

(hereafter we shall set $B_{\mu\nu} = 0$ in order to remove the effective three-dimensional cosmological constant from our consideration).

c) vector fields $A_\mu^{(a)} = ((A_1)_\mu^p, (A_2)_\mu^{p+d}, (A_3)_\mu^{2d+I})$ ($a = 1, \dots, 2d + n$)

$$\begin{aligned} (A_1)_\mu^p &= \frac{1}{2} G^{pq} G_{q+3,\mu}^{(D)}, & (A_3)_\mu^{I+2d} &= -\frac{1}{2} A_\mu^{(D)I} + A_q^I A_\mu^q, \\ (A_2)_\mu^{p+d} &= \frac{1}{2} B_{p+3,\mu}^{(D)} - B_{pq} A_\mu^q + \frac{1}{2} A_p^I A_\mu^{I+2d}. \end{aligned} \quad (39)$$

In three dimensions all vector fields $A_\mu^{(a)}$, can be dualized on-shell with the aid of the pseudoscalar potentials u, v and s in the following form:

$$\begin{aligned} \nabla \times \vec{A}_1 &= \frac{1}{2} e^{2\phi} G^{-1} \left(\nabla u + (B + \frac{1}{2} A A^T) \nabla v + A \nabla s \right), \\ \nabla \times \vec{A}_3 &= \frac{1}{2} e^{2\phi} (\nabla s + A^T \nabla v) + A^T \nabla \times \vec{A}_1, \\ \nabla \times \vec{A}_2 &= \frac{1}{2} e^{2\phi} G \nabla v - (B + \frac{1}{2} A A^T) \nabla \times \vec{A}_1 + A \nabla \times \vec{A}_3. \end{aligned} \quad (40)$$

Thus, the resulting effective three-dimensional theory describes the scalars G, B, A and ϕ and the pseudoscalars u, v and s coupled to the metric $g_{\mu\nu}$.

We further define the so-called *matrix* Ernst potentials (MEP) from all these scalar and pseudoscalar potentials in order to express the low-energy effective action of the heterotic string in a similar form to the formulation of the stationary Einstein–Maxwell theory in terms of the complex Ernst potentials [6]:

$$\mathcal{X} = \begin{pmatrix} -e^{-2\phi} + v^T X v + v^T A s + \frac{1}{2} s^T s & v^T X - u^T \\ X v + u + A s & X \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} s^T + v^T A \\ A \end{pmatrix}, \quad (41)$$

where $X = G + B + \frac{1}{2} A A^T$. These potentials are of dimensions $(d+1) \times (d+1)$ and $(d+1) \times n$, respectively.

The physical meaning of their components are as follows: The relevant information about the gravitational field is encoded in the potential X , while its rotational nature is parameterized by the pseudoscalar u ; ϕ is the dilatonic field; v is related to the multi-dimensional components of the antisymmetric tensor field of Kalb–Ramond. Finally, A and s represent electric and magnetic potentials.

3.1 Stationary effective action of heterotic string and field equations in the language of MEP

In terms of MEP the effective three-dimensional theory adopts the form [6]:

$${}^3\mathcal{S} = \int d^3x \, |g|^{\frac{1}{2}} \{ -{}^3R + {}^3\mathcal{L}_{HS} \}, \quad (42)$$

where the matter Lagrangian is given by

$$\mathcal{L}_{HS} = \text{Tr} \left[\frac{1}{4} (\nabla \mathcal{X} - \nabla \mathcal{A} \mathcal{A}^T) \mathcal{G}^{-1} (\nabla \mathcal{X}^T - \mathcal{A} \nabla \mathcal{A}^T) \mathcal{G}^{-1} + \frac{1}{2} \nabla \mathcal{A}^T \mathcal{G}^{-1} \nabla \mathcal{A} \right], \quad (43)$$

3R is the three-dimensional curvature scalar and the matrix potential \mathcal{X} is defined by $\mathcal{X} = \mathcal{G} + \mathcal{B} + \frac{1}{2} \mathcal{A} \mathcal{A}^T$.

The symmetric part of the potential is given by the matrix $\mathcal{G} = \frac{1}{2} (\mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T)$ and the antisymmetric one by $\mathcal{B} = \frac{1}{2} (\mathcal{X} - \mathcal{X}^T)$; these matrices are parameterized as follows:

$$\mathcal{G} = \begin{pmatrix} -e^{-2\phi} + v^T G v & v^T G \\ G v & G \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 0 & v^T B - u^T \\ B v + u & B \end{pmatrix}. \quad (44)$$

By making use of the conventional method of variations, from the effective action (42) one obtains both the *Einstein equations*

$${}^3R_{\mu\nu} = \text{Tr} \left[\frac{1}{4} (\nabla_\mu \mathcal{X} - \nabla_\mu \mathcal{A} \mathcal{A}^T) \mathcal{G}^{-1} (\nabla_\nu \mathcal{X}^T - \mathcal{A} \nabla_\nu \mathcal{A}^T) \mathcal{G}^{-1} + \frac{1}{2} \nabla_\mu \mathcal{A}^T \mathcal{G}^{-1} \nabla_\nu \mathcal{A} \right], \quad (45)$$

as well as the *Ernst equations* for the potentials \mathcal{X} and \mathcal{A} which represent the matter sector of the theory:

$$\nabla^2 \mathcal{X} - 2 (\nabla \mathcal{X} - \nabla \mathcal{A} \mathcal{A}^T) (\mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T)^{-1} \nabla \mathcal{X} = 0,$$

$$\nabla^2 \mathcal{A} - 2 (\nabla \mathcal{X} - \nabla \mathcal{A} \mathcal{A}^T) (\mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T)^{-1} \nabla \mathcal{A} = 0,$$

as a matrix version of the equations of the stationary Einstein–Maxwell theory.

As we have pointed out above, these differential equations are not so simple to solve in a closed form. However, one can make use of the similarity which exists with respect to the equations of the stationary Einstein–Maxwell theory in order to guess and write down the solutions in a direct way or to perform nonlinear symmetries to generate new exact solutions from known ones (for some examples see [14]).

4 Heterotic string *vs.* Einstein–Maxwell

Thus, it has been shown that there exists a close relation between the stationary effective actions of the heterotic string and the Einstein–Maxwell theory:

$$\mathcal{X} \longleftrightarrow -E, \quad \mathcal{A} \longleftrightarrow F, \quad (46)$$

$$\text{matrix transposition} \quad \longleftrightarrow \quad \text{complex conjugation}.$$

One can realize that the relation (46) allows us to generalize in a straightforward way the results obtained within the framework of the Einstein–Maxwell theory to the realm of the heterotic string (where a suitable physical interpretation will be needed since more fields are involved) by making use of the MEP formalism. Actually, the four–dimensional Einstein–Maxwell theory, being reduced to three dimensions, can be written as a special case of the MEP formalism with some peculiarities in terms of the complex Ernst potentials E and F [9].

Let us rewrite them in a less conventional form

$$-\mathcal{X}_{EM} = \text{Re}E + \sigma_2 \text{Im}E, \quad \mathcal{A}_{EM} = \text{Re}F + \sigma_2 \text{Im}F, \quad \text{where} \quad \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (47)$$

We can treat these matrices as the matrix Ernst potentials (41) of the $D = 4$ theory (35) with $\phi^{(4)} = B_{MN}^{(4)} = 0$. Then we conclude that we need two Abelian gauge fields $n = 2$ and that they should satisfy the following constraint

$$s^1 = A^2 = \text{Re}F, \quad -s^2 = A^1 = \text{Im}F. \quad (48)$$

Note, that s^I ($I = 1, 2$) describe the magnetic potentials, whereas A^I are the electric ones. Thus, both Maxwell fields arising in the framework of the representation (41)–(43) and (47) turn out to be mutually conjugated (i.e. $F_{MN}^{(4)2} = \tilde{F}_{MN}^{(4)1}$ in four dimensions). Next, for the single extra metric component one has:

$$G = -\frac{1}{2}(E + E^* + FF^*) \equiv f, \quad \text{and} \quad u = \text{Im}E. \quad (49)$$

By taking into account that $\mathcal{G} = G$, and by substituting equations (47) and (49) into the matter Lagrangian (43), we obtain

$$\mathcal{L}_{EM} = \frac{1}{2f^2} |\nabla E + F^* \nabla F|^2 - f^{-1} |\nabla F|^2. \quad (50)$$

As we already have seen, this is precisely the matter Lagrangian of the stationary Einstein–Maxwell theory. Thus, our MEP formulation of the heterotic string theory includes the Einstein–Maxwell theory as a special case.

It is worth noticing as well that the higher dimensional General Relativity theory can also be written in terms of a matrix Ernst potential when reduced to three dimensions. This fact corresponds to a special case in which the matter degrees of freedom of the low–energy heterotic string theory (35) vanish: the anti–symmetric Kalb–Ramond tensor field $B_{MN}^{(D)} = 0$, the dilaton $\phi^{(D)} = 0$ and the Abelian gauge fields $A_M^{(D)I} = 0$, so that the matrix Ernst potential is symmetric $\mathcal{X} = \mathcal{G}$ and $\mathcal{B} = \mathcal{A} = 0$. It should also be mentioned that the three-dimensional dilaton field must remain nontrivial since it is identified with the determinant of the extra dimensional metric according to the definitions (36) and (37).

Thus, this parametrization of the above mentioned higher dimensional theories in terms of the MEP can be very useful when performing a complete classification of the higher dimensional ($D \geq 5$) black objects (holes, rings, Saturns, etc.) obtained in the literature during last years (see [10] for a review).

5 Nonlinear hidden symmetries and their possible applications in $D \geq 5$

One of the advantages of the (matrix) Ernst potential formalism is that the study of symmetries (conservation laws) of the stationary effective action can be performed in a very straightforward way. It turns out that the complete symmetry group, apart from rescalings and shifts of the Ernst potentials, involves nonlinear symmetries that were initially called *hidden* in the framework of General Relativity; moreover, an infinite-dimensional double hidden symmetry structure was revealed for string effective actions [15]. In particular, these symmetries act nontrivially in the charge space of a seed solution and can be used to generate new charged solutions from uncharged ones. There also other effects when applying this symmetries (see, for instance, [4, 5, 9, 16, 17]).

Here we shall quote just the symmetries which preserve the asymptotic properties of the (matrix) Ernst potentials for physically meaningful field configurations of both the stationary Einstein–Maxwell and low–energy heterotic string theories. These symmetries possess the same form for both theories and allow one to generate similar solutions in both realms [9].

For the stationary Einstein–Maxwell theory we have:

$$E \rightarrow E, \quad F \rightarrow e^{i\alpha} F; \quad (\text{EMT}) \quad (51)$$

$$E \rightarrow \frac{E + i\epsilon}{1 + i\epsilon E}, \quad F \rightarrow \frac{1 - i\epsilon}{1 + i\epsilon E} F; \quad (\text{NET}) \quad (52)$$

$$E \rightarrow \frac{E + \frac{1}{2} |\lambda_{\mathcal{H}}|^2 - \bar{\lambda}_{\mathcal{H}} F}{1 - \bar{\lambda}_{\mathcal{H}} F + \frac{1}{2} |\lambda_{\mathcal{H}}|^2 E}, \quad F \rightarrow \frac{\left(1 + \frac{1}{2} |\lambda_{\mathcal{H}}|^2\right) F - \lambda_{\mathcal{H}} (E + 1)}{1 - \bar{\lambda}_{\mathcal{H}} F + \frac{1}{2} |\lambda_{\mathcal{H}}|^2 E}, \quad (\text{NHT}) \quad (53)$$

where EMT stands for Electric–Magnetic Transformation, NET for Normalized Ehlers Transformation and NHT for Normalized Harrison Transformation, the parameter $\lambda_{\mathcal{H}}$ is complex while the parameters α and ϵ are real. It is easy to check that when the parameters $\lambda_{\mathcal{H}}$, α and ϵ vanish, one recovers the original (seed) potentials.

On the other hand, for the stationary low–energy effective action of the heterotic string we have the following matrix symmetries:

$$\mathcal{X} \rightarrow \mathcal{X} + \lambda_{\mathcal{X}}, \quad \mathcal{A} \rightarrow \mathcal{A} \quad \text{with} \quad \lambda_{\mathcal{X}}^T = -\lambda_{\mathcal{X}} \quad (54)$$

$$\mathcal{A} \rightarrow \mathcal{A} + \lambda_{\mathcal{A}}, \quad \mathcal{X} \rightarrow \mathcal{X} + \mathcal{A} \lambda_{\mathcal{A}}^T + \frac{1}{2} \lambda_{\mathcal{A}} \lambda_{\mathcal{A}}^T \quad (55)$$

$$\mathcal{A} \rightarrow \mathcal{A} \mathcal{T}, \quad \mathcal{X} \rightarrow \mathcal{X}, \quad \text{where} \quad \mathcal{T} \mathcal{T}^T = 1 \quad (56)$$

$$\mathcal{X} \rightarrow \mathcal{S}^T \mathcal{X} \mathcal{S}, \quad \mathcal{A} \rightarrow \mathcal{S}^T \mathcal{A}, \quad \text{with} \quad \mathcal{S} \rightarrow (\mathcal{S}^T)^{-1}. \quad (57)$$

$$\begin{aligned} \mathcal{A} &\rightarrow (1 + \Sigma \lambda_{\mathcal{E}}) (1 + \mathcal{X} \lambda_{\mathcal{E}})^{-1} \mathcal{A}, \\ \mathcal{X} &\rightarrow (1 + \Sigma \lambda_{\mathcal{E}}) (1 + \mathcal{X} \lambda_{\mathcal{E}})^{-1} \mathcal{X} (1 - \lambda_{\mathcal{E}} \Sigma) + \Sigma \lambda_{\mathcal{E}} \Sigma. \end{aligned} \quad (\text{NET}) \quad (58)$$

$$\begin{aligned} \mathcal{A} &\rightarrow \left(1 + \frac{1}{2} \Sigma \lambda_{\mathcal{H}} \lambda_{\mathcal{H}}^T\right) \left(1 - \mathcal{A} \lambda_{\mathcal{H}}^T + \frac{1}{2} \mathcal{X} \lambda_{\mathcal{H}} \lambda_{\mathcal{H}}^T\right)^{-1} \times \\ &\quad (A - \mathcal{X} \lambda_{\mathcal{H}}) + \Sigma \lambda_{\mathcal{H}}, \quad (\text{NHT}) \\ \mathcal{X} &\rightarrow \left(1 + \frac{1}{2} \Sigma \lambda_{\mathcal{H}} \lambda_{\mathcal{H}}^T\right) \left(1 - \mathcal{A} \lambda_{\mathcal{H}}^T + \frac{1}{2} \mathcal{X} \lambda_{\mathcal{H}} \lambda_{\mathcal{H}}^T\right)^{-1} \times \\ &\quad \left[\mathcal{X} + \left(\mathcal{A} - \frac{1}{2} \mathcal{X} \lambda_{\mathcal{H}}\right) \lambda_{\mathcal{H}}^T \Sigma\right] + \frac{1}{2} \Sigma \lambda_{\mathcal{H}} \lambda_{\mathcal{H}}^T \Sigma. \end{aligned} \quad (59)$$

where $\lambda_{\mathcal{E}}^T = -\lambda_{\mathcal{E}}$ and $\lambda_{\mathcal{H}}$ is a real rectangular matrix of dimension $(d+1) \times n$.

The last pair of nonlinear symmetries can be applied to construct new exact solutions starting from known (sometimes quite simple) field configurations in both theories. As an example one can cite the construction of the of the Reissner–Nordström solution starting from the Schwarzschild black hole one in the 4D Einstein–Maxwell theory.

We finally quote a procedure to construct new charged field configurations from known neutral solutions within the framework of theories like General Relativity and the effective low–energy action of the heterotic string with more than four dimensions (in the spirit of [16, 17]). Thus, this procedure can be applied to the construction of charged black holes, black rings and black Saturns if $D = 5$, and charged multiple black rings in $D = 6$:

1. Write the exact solution of the uncharged field configuration (black ring or black Saturn, for instance) in the form of a generalized Weyl metric [18, 19] by making use of a suitable coordinate system.
2. Identify the symmetric and antisymmetric parts of the matrix Ernst potential \mathcal{X} .
3. Perform the nonlinear hidden symmetry NHT on the matrix Ernst potentials \mathcal{X} and \mathcal{A} .
4. Write the new higher–dimensional charged exact solution with the aid of \mathcal{X} and \mathcal{A} .
5. Physically interpret the new solution with the aid of the behaviour of the fields and their properties.

This procedure can be performed also in a wider class of higher–dimensional field configurations that have the form of a stationary axisymmetric seed solution (the so–called Weyl–Papapetrou class) [20] and it is interesting to see what kind of physical configurations arise after applying the MEP symmetry method.

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References

- [1] F.J. Ernst, Phys. Rev. **168** 1415 (1968); **167** 1175 (1968); J. Math. Phys. **12** 2395 (1971).
- [2] W. Israel and G.A. Wilson, J. Math. Phys. **13** 865 (1972).
- [3] D. Kramer, H. Stephani, M. McCallum and E. Herlt, *Exact Solutions of the Einstein's Field Equations*, (Deutcher Verlag der Wissenschaften, Berlin, 1980).
- [4] W. Kinnersley, J. Math. Phys. **18** 1529 (1977); W. Kinnersley and D.M. Chitre, J. Math. Phys. **18** 1538 (1977); **19** 1926 (1978); Phys. Rev. Lett. **40** 1608 (1978).
- [5] D.V. Gal'tsov and O. Kechkin, Phys. Rev. **D50** 7394 (1994); D. Gal'tsov, A.A. García, and O. Kechkin, J. Math. Phys. **36** 5023 (1995).
- [6] A. Herrera–Aguilar and O. Kechkin, Int. J. Mod. Phys. **A12** 1573 (1997); **A13** 393 (1998); **A14** 1345 (1999); N. Barbosa–Cendejas and A. Herrera–Aguilar, Gen. Rel. Grav. **35** 449 (2003).
- [7] V. Belinski and E. Verdaguer, *Gravitational Solitons*, (Cambridge University Press, Cambridge, 2001).
- [8] A. Herrera–Aguilar and O. Kechkin, Mod. Phys. Lett. **A13** 1907 (1998).
- [9] A. Herrera–Aguilar and O. Kechkin, Phys. Rev. **D59** 124006 (1999).
- [10] R. Emparan and H.S. Reall, Phys. Rev. Lett. **88** 4877 (2002); Class. Quant. Grav. **23** R169 (2006); Living Rev. Rel. **11** 6 (2008); R. Emparan and P. Figueras, JHEP **1011** 022 (2010).
- [11] H. Elvang and P. Figueras, JHEP **0705** 050 (2007).
- [12] N. Marcus and J.H. Schwarz, Nucl. Phys. **B228** 145 (1983); J. Maharana and J.H. Schwarz, Nucl. Phys. **B390** 3 (1993).
- [13] A. Sen, Nucl. Phys. **B434** 179 (1995).

- [14] A. Herrera–Aguilar and O. Kechkin, *Mod. Phys. Lett.* **A13** 1629 (1998); 1979 (1998); *Class. Quant. Grav.* **16** 1745 (1999); *Mod. Phys. Lett.* **A16** 29 (2001); *Int. J. Mod. Phys.* **A17** 2485 (2002); *J. Math. Phys.* **45** 216 (2004); R. Becerril and A. Herrera–Aguilar, *J. Math. Phys.* **46** 052503 (2005); A. Herrera–Aguilar and M. Nowakowski, *Class. Quant. Grav.* **21** 1015 (2004).
- [15] Y.–J. Gao, *Gen. Rel. Grav.* **35** 1573 (2003); *Phys. Rev.* **D76** 044025 (2007); *Rept. Math. Phys.* **62** 1 (2008); *Mod. Phys. Lett.* **A24** 311 (2009).
- [16] A. Herrera–Aguilar and H.R. Márquez–Falcon, in *ICHEP 2006: Proceedings, 2 Vols.* Eds. A. Sissakian, G. Kozlov and E. Kolganova, (Singapore, World Scientific, 2007).
- [17] A. Herrera–Aguilar, *Rev. Mex. Fis.* **49S2** 141 (2003); *Mod. Phys. Lett.* **A19** 2299 (2004); A. Herrera–Aguilar, J.O. Téllez–Vázquez and J.E. Paschalis, *Regular Chaot. Dyn.* **14** 526 (2009).
- [18] R. Emparan and H.S. Reall, *Phys. Rev.* **D65** 084025 (2002).
- [19] T. Harmark, *Phys. Rev.* **D70** 124002 (2004).
- [20] G.T. Horowitz and A. Sen, *Phys. Rev.* **D54** 808 (1996); M. Cvetič and D. Youm, *Nucl. Phys.* **B476** 118 (1996); A.A. Tseytlin, *Mod. Phys. Lett. A* **11** 689 (1996); J.C. Breckenridge, D.A. Lowe, R.C. Myers, A.W. Peet, A. Strominger and C. Vafa, *Phys. Lett.* **B381** 423 (1996); R. Kallosh, A. Rajaraman and W.K. Wong, *Phys. Rev.* **D55** 3246 (1997); J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, *Phys. Lett.* **B391** 93 (1997); C.A.R. Herdeiro, *Nucl. Phys.* **B582** 363 (2000); T. Matos, D. Núñez, G. Estévez and M. Ríos, *Gen. Rel. Grav.* **32** 1499 (2000); H. Elvang, *Phys. Rev.* **D68** 124016 (2003); A. Herrera–Aguilar and R.R. Mora–Luna, *Phys. Rev.* **D69** 105002 (2004); S.S. Yazadjiev, *Phys. Rev.* **D72** 104014 (2005); *Class. Quant. Grav.* **22** 3875 (2005); *Phys. Rev.* **D73** 064008 (2006); *Phys. Rev.* **D73** 124032 (2006); *Phys. Rev.* **D77** 127501 (2008).